JEE ADV - PHYSICS

DPP -03

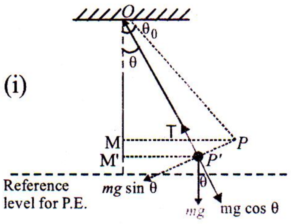
3 Feb 2020

TOPICS : ROTATION MOTION (SOLUTION)

1.

$$T - mg\cos\theta = \frac{mv^2}{\ell}$$

$$\therefore \quad T = \frac{mv^2}{\ell} + mg\cos\theta \; \text{... (i)}$$



In ΔOPM, $\cos\theta_0 = \frac{OM}{\ell}$

$\Rightarrow OM = \ell\cos\theta_0$

In $\Delta OP'M'$, $\cos\theta = \frac{OM'}{\ell}$

$\Rightarrow OM' = \ell\cos\theta$

$OM' - OM = \ell(\cos\theta - \cos\theta_0)$

Loss in potential energy = Gain in kinetic energy (Activity P to P')

$\Rightarrow mg\ell(\cos\theta - \cos\theta_0) = \frac{1}{2}mv^2$

$\Rightarrow v^2 = 2g\ell(\cos\theta - \cos\theta_0)$ ...(ii)

From (i) and (ii)

$$T = \frac{m}{\ell} \times 2g\ell(\cos\theta - \cos\theta_0) + mg\cos\theta$$

$\therefore \quad T = 3mg\cos\theta - 2mg\cos\theta_0$

$\Rightarrow \quad T = mg(3\cos\theta - 2\cos\theta_0)$

From equation (i) it is clear that the tension is maximum when $\cos\theta = 1$ i.e., $\theta = 0°$

$\therefore T = mg$

Hence, $T_{max} = \frac{mv^2}{\ell} + mg$ ... (iii)

From eqn. (ii)

$v^2 = 2g\ell(1 - \cos\theta_0)$ ... (iv)

From (iii) and (iv)

$$T_{max} = \frac{m}{\ell}[2g\ell(1 - \cos\theta_0)] + mg$$

$\therefore \quad T_{max} = 3mg - 2mg\cos\theta_0$

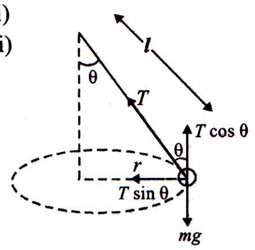
$80 = 3 \times 40 - 2 \times 40\cos\theta_0$

$\Rightarrow \quad 80\cos\theta_0 = 40 \Rightarrow \cos\theta_0 = \frac{1}{2} \Rightarrow \theta_0 = 30°$

2. Suppose mass m moves around a circular path of radius r. Let the string makes an angle θ with the vertical. Resolving tension T, we get

and, $T\sin\theta = mr\omega^2$ ... (i)

$T\cos\theta = mg$ ... (ii)



$\therefore \quad \tan\theta = \frac{r\omega^2}{g}$

From diagram, $\sin\theta = \frac{r}{\ell}$

$\Rightarrow \quad r = \ell\sin\theta$

$\therefore \quad \tan\theta = \ell\sin\theta\frac{\omega^2}{g}$

$\omega^2 = \frac{\tan\theta . g}{\ell\sin\theta} \qquad \omega = \sqrt{\frac{g}{\ell\cos\theta}}$

$\Rightarrow \nu = \frac{1}{2\pi}\sqrt{\frac{g}{\ell\cos\theta}}$ ... (iii)

From (ii), $T\cos\theta = mg$.

For M to remain stationary, $T = Mg$

$\therefore \quad Mg\cos\theta = mg$

$\Rightarrow \quad \cos\theta = \frac{m}{M}$ ... (iv)

From (iii) and (iv), $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}\frac{M}{m}}$

3. Let σ be the mass per unit area.

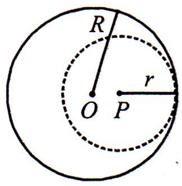
Then the mass of the whole disc $= \sigma \times \pi R^2$

Mass of the portion removed $= \sigma \times \pi r^2$

$R = 28$ cm; $r = 21$ cm; $OP = 7$ cm

Taking O as the origin

The position of c.m.



$$x = \frac{m_1x_1 - m_2x_2}{m_1 - m_2}$$

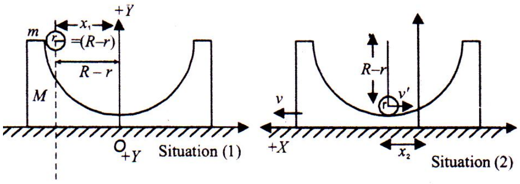
$$= \frac{\sigma \times \pi R^2(0) - \sigma \times \pi r^2 \times 7}{\sigma\pi R^2 - \sigma\pi r^2}$$

$$= \frac{-(21)^2 \times 7}{(28)^2 - (21)^2} = -\frac{21 \times 21 \times 7}{7 \times 49} = -9 \text{ cm}$$

This means that the c.m. lies at a distance of 9 cm from the origin towards left.

4. C.M. of the system of two bodies in situation (i) in x-coordinate

$$x_C = \frac{M \times 0 + mx_1}{M+m} = \frac{mx_1}{M+m} \quad \text{...(i)}$$



C.M. of the system in situation (ii) in x-coordinate is

$$x'_C = \frac{M \times x_2 + m \times x_2}{M+m} = x_2 \quad \text{...(ii)}$$

Since no external force is in x-direction

$\therefore \quad x_C = x'_C$

$$\therefore \quad x_2 = \frac{mx_1}{M+m} = \frac{m(R-r)}{M+m}$$

Applying conservation of linear momentum,
Initial Momentum = Final Momentum

$0 = MV - mv$

$$\therefore \quad v = \frac{MV}{m} \quad \text{....(iii)}$$

Applying the concept of conservation of energy, we get
Loss in P.E. of mass m = Gain in K.E. of mass M and Gain in K.E. of mass m

$$\Rightarrow \quad mg(R-r) = \frac{1}{2}MV^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \quad 2mg(R-r) = MV^2 + m\frac{M^2V^2}{m^2} \quad \text{[from (iii)]}$$

$$\Rightarrow \quad 2mg(R-r) = MV^2 + \frac{M^2V^2}{m}$$

$$2mg(R-r) = MV^2\left[1 + \frac{M}{m}\right] = MV^2\left[\frac{m+M}{m}\right]$$

$$\Rightarrow \quad \frac{2m^2 g(R-r)}{M(m+M)} = V^2 \quad \Rightarrow \quad V = m\sqrt{\frac{2g(R-r)}{M(m+M)}}$$

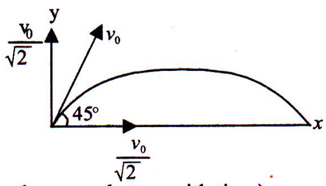
5. The angular momentum is given by $L = xp_y - yp_x$
$= m[xv_y - yv_x]$

(x, y) are the coordinates of the particle after time $t = \dfrac{v_0}{g}$ and

v_x, v_y are the components of velocities at that time.

For v_x and v_y

$$v_x = v_0 \cos 45^\circ = \frac{v_0}{\sqrt{2}}$$



(The horizontal velocity does not change with time)
Applying $v = u + at$ in the vertical direction to find v_y

$$v_y = (v_0 \sin 45^\circ) - g\left(\frac{v_0}{g}\right) = \frac{v_0}{\sqrt{2}} - g \times \frac{v_0}{g} = \frac{v_0}{\sqrt{2}} - v_0$$

For x and y

In horizontal direction $x = v_x \times t$

$$\therefore \quad x = \frac{v_0}{\sqrt{2}} \times \frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}g}$$

In vertical direction applying $S = ut + \frac{1}{2}at^2$

$$y = \frac{v_0}{\sqrt{2}} \times \frac{v_0}{g} - \frac{1}{2}g\frac{v_0^2}{g^2} = \frac{v_0^2}{\sqrt{2}g} - \frac{v_0^2}{2g}$$

Putting the values in the above equation

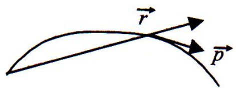
$$L = m\left[\frac{v_0^2}{\sqrt{2}g} \times \left(\frac{v_0}{\sqrt{2}} - v_0\right) - \left(\frac{v_0^2}{\sqrt{2}g} - \frac{v_0^2}{2g}\right)\frac{v_0}{\sqrt{2}}\right]$$

$$L = m\left[\frac{v_0^3}{2g} - \frac{v_0^3}{\sqrt{2}g} - \frac{v_0^3}{2g} + \frac{v_0^3}{2\sqrt{2}\,g}\right]$$

$$L = \frac{mv_0^3}{g}\left[\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right] \qquad L = \frac{-mv_0^3}{2\sqrt{2}\,g}$$

Now, $\vec{L} = \vec{r} \times \vec{p}$

Note : The direction of L is perpendicular to the plane of motion and is directed away from the reader.



6. **KEY CONCEPT :** Applying law of conservation of energy at point D and point A
$P.E.$ at $D = P.E.$ at $Q + (K.E.)_T + (K.E.)_R$ where
$(K.E.)_T$ = Translational $K.E.$ and $(K.E.)_R$ = Rotational $K.E.$

$$\Rightarrow \quad mg(2.4) = mg(1) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad \text{...(i)}$$

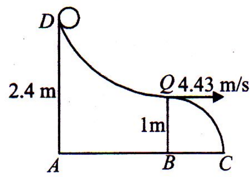
Since the case is of rolling without slipping

$\therefore \quad v = r\omega$

$\therefore \quad \omega = \dfrac{v}{r}$ where r is the radius of the sphere

Also, $\quad I = \dfrac{2}{5}mr^2$

Putting in equation (i)



$$mg(2.4-1) = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\frac{v^2}{r^2}$$

or, $g \times 1.4 = \frac{7v^2}{10} \Rightarrow v = 4.43$ m/s

After point Q, the body takes a parabolic path.
The vertical motion parameters of parabolic motion will be

$u_y = 0$ $\quad S_y = 1$m

$a_y = 9.8$ m/s^2 $\quad t_y = ?$

$\therefore \quad S = ut + \frac{1}{2}at^2 \Rightarrow 1 = 4.9\,t_y^2$

$t_y = \frac{1}{\sqrt{4.9}} = 0.45$ sec

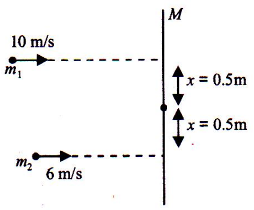
Applying this time in horizontal motion of parabolic path,
$BC = 4.43 \times 0.45 = 2$m

Note : During its flight as a projectile, the sphere continues to rotate because of conservation of angular momentum.

7. Initial Kinetic Energy

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}MV^2$$

$$= \frac{1}{2}0.08 \times 10^2 + \frac{1}{2}0.08 \times 6^2 + 0 = 5.44\,\text{J} \quad \text{...(i)}$$



Applying law of conservation of linear momentum during collision

$m_1 \times v_1 + m_2 \times v_2 = (M + m_1 + m_2)\,V_c$

where V_c is the velocity of centre of mass of the bar and particles sticked on it after collision

$0.08 \times 10 + 0.08 \times 6 = (0.16 + 0.08 + 0.08)\,V_c$

$\Rightarrow \quad V_c = 4$ m/s

$\therefore \quad$ Translational kinetic energy after collision

$$= \frac{1}{2}(M + m_1 + m_2)V_c^2 = 2.56\,\text{J} \quad \text{...(ii)}$$

Applying conservation of angular momentum of the bar and two particle system about the centre of the bar.

Since external torque is zero, the initial angular momentum is equal to final angular momentum.

Initial angular momentum

$= m_1v_1 \times x - m_2v_2x$

$= 0.08 \times 10 \times 0.5 - 0.08 \times 6 \times 0.5$

$= 0.4 - 0.24 = 0.16$ kg m^{2}s^{-1} (In clockwise direction)

Final angular momentum $= I\omega$

$$= \left[\frac{M\ell^2}{12} + m_1x^2 + m_2x^2\right]\omega$$

$$= \left[\frac{(0.16)(\sqrt{3})^2}{12} + 0.08 \times (0.5)^2 + (0.08)(0.5)^2\right]\omega$$

$= 0.08\,\omega$

$\therefore \quad 0.08\,\omega = 0.16 \Rightarrow \omega = 2$ rad/s ...(iii)

The rotational kinetic energy

$$= \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.08 \times 2^2 = 0.16\,\text{J} \quad \text{...(iv)}$$

The final kinetic energy

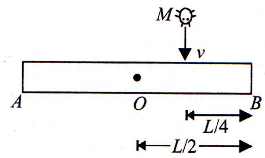
$= $ Translational $K.E.$ + Rotational $K.E.$

$= 2.56 + 0.16 = 2.72$ J

The change in $K.E.$ = Initial $K.E.$ – Final $K.E.$

$= 5.44 - 2.72 = 2.72$ J

8. **(a)** Let us consider the system of homogeneous rod and insect and apply conservation of angular momentum during collision about the point O.



Angular momentum of the system before collision = angular momentum of the system after collision.

$$Mv \times \frac{L}{4} = I\omega$$

Where I is the moment of inertia of the system just after collision and ω is the angular velocity just after collision.

$$\Rightarrow \quad Mv\frac{L}{4} = \left[M\left(\frac{L}{4}\right)^2 + \frac{1}{12}ML^2\right]\omega$$

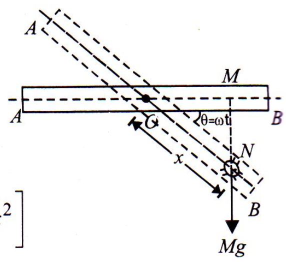
$$\Rightarrow \quad Mv \times \frac{L}{4} = \frac{ML^2}{4}\left[\frac{1}{4} + \frac{1}{3}\right]\omega = \frac{ML^2}{4}\left[\frac{3+4}{12}\right]$$

$$= \frac{ML^2}{4} \times \frac{7}{12} \times \omega \Rightarrow \quad \omega = \frac{12}{7}\frac{v}{L}.$$

(b) Note : Initially the torque due to mass OB of the rod (acting in clockwise direction) was balanced by the torque due to mass OA of the rod (acting in anticlockwise direction). But after collision there is an extra mass M of the insect which creates a torque in the clockwise direction, which tends to create angular acceleration in the rod. But the same is compensated by the movement of insect towards B due to which moment of inertia I of the system increases.

Let at any instant of time t the insect be at a distance x from the centre of the rod and the rod has turned through an angle θ $(= \omega t)$ w.r.t its original position.

Instantaneous torque,



$$\tau = \frac{dL}{dt} = \frac{d}{dt}(I\omega)$$

$$= \omega\frac{dI}{dt}$$

$$= \omega\frac{d}{dt}\left[\frac{1}{12}ML^2 + Mx^2\right]$$

$$= 2M\omega x\frac{dx}{dt} \quad \text{...(i)}$$

This torque is balanced by the torque due to weight of insect.

τ = Force × Perpendicular distance of force with axis of rotation $= Mg \times (OM)$

$= Mg\,(x\cos\theta)$ ... (ii)

From (i) and (ii)

$$2M\omega x\frac{dx}{dt} = Mg\,(x\cos\theta) \Rightarrow dx = \left(\frac{g}{2\omega}\right)\cos\omega t\,dt$$

On integration, taking limits

$$\int_{L/4}^{L/2} dx = \frac{g}{2\omega}\int_0^{\pi/2\omega}\cos\omega t\,dt$$

when $x = \frac{L}{4}, \omega t = 0$

$$[x]_{L/4}^{L/2} = \frac{g}{2\omega^2}[\sin\omega t]_0^{\pi/2\omega}$$

when $x = \frac{L}{2}, \omega t = \frac{\pi}{2}$

$$\Rightarrow \left(\frac{L}{2}-\frac{L}{4}\right) = \frac{g}{2\omega^2}\left[\sin\frac{\pi}{2}-\sin 0\right]$$

$$\Rightarrow \frac{L}{4} = \frac{g}{2\omega^2} \Rightarrow \omega = \sqrt{\frac{2g}{L}}$$

But $\omega = \frac{12}{7}\frac{v}{L} \Rightarrow \frac{12}{7}\frac{v}{L} = \sqrt{\frac{2g}{L}} \Rightarrow v = \frac{7}{12}\sqrt{2gL}$

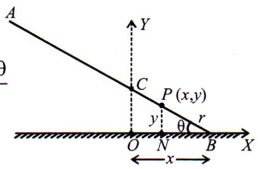
$$\Rightarrow v = \frac{7}{12}\sqrt{2\times 10\times 1.8} = 3.5\ \text{ms}^{-1}$$

9. **(i)** Initially, the rod stands vertical. *A* straight disturbance makes the rod to rotate. While rotating, the force acting on the rod are its weight and normal reaction. These forces are vertical forces and cannot create a horizontal motion. Therefore the centre of mass of the rod does not move horizontally. The center of mass moves vertically downwards. Thus the path of the center of mass is a straight line.

(ii) Trajectory of an arbitrary point of the rod

Consider an arbitrary point *P* on the rod located at (x, y) and at a distance r from the end *B*. Let θ be the angle of inclination of the rod with the horizontal at this position.

In ΔBNP, $\sin\theta = \frac{y}{r}$ ... (i)



$$\cos\theta = \frac{x+BN}{L/2} = \frac{x+r\cos\theta}{L/2}$$

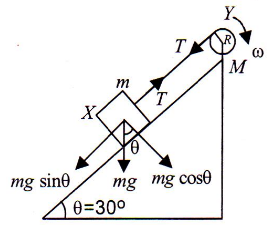
$$\Rightarrow \cos\theta = \frac{x}{\frac{L}{2}-r} \quad \text{... (ii)}$$

From (i) and (ii) $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \frac{y^2}{r^2} + \frac{x^2}{\left(\frac{L}{2}-r\right)^2} = 1$$

This is equation of an ellipse.

10. **(i)** The drum is given an initial velocity such that the block *X* starts moving up the plane.



As the time passes, the velocity of the block decreases. The linear retardation a, of the block X is given by

$mg\sin\theta - T = ma$ ... (i)

The linear retardation of the block and the angular acceleration of the drum (α) are related as

$a = R\alpha$ ... (ii)

where R is the radius of the drum.

The retarding torque of the drum is due to tension T in the string.

$\tau = T\times R$

But $\tau = I\alpha$. where $I = M.I.$ of drum about its axis of rotation.

$\therefore\ T\times R = \frac{1}{2}MR^2\alpha$ ... (iii) $\left[\because I = \frac{1}{2}MR^2\right]$

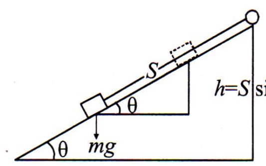
From (ii), $TR = \frac{1}{2}MR^2\frac{a}{R} \Rightarrow a = \frac{2T}{M}$

Substituting this value in (i)

$$mg\sin\theta - T = m\times\frac{2T}{M} \Rightarrow mg\sin\theta = \left(1+\frac{2m}{M}\right)T$$

$$\therefore\ T = \frac{(mg\sin\theta)\times M}{M+2m} = \frac{0.5\times 9.8\times\sin 30°\times 2}{2+2\times 0.5} = 1.63\ \text{N}$$

(ii) The total kinetic energy of the drum and the block at the instant when the drum is having angular velocity 10 rads^{-1} gets converted into the potential energy of the block.



$$[(K.E.)_{Rotational}]_{drum} + \{(K.E.)_{Translational}]_{block} = mgh$$

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mgS\sin\theta$$

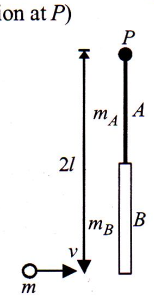
$$\frac{1}{2}I\omega^2 + \frac{1}{2}m(R\omega)^2 = mgS\sin\theta \quad [\because v = R\omega]$$

$$\Rightarrow \frac{1}{2}MR^2\omega^2 + \frac{1}{2}mR^2\omega^2 = mgS\sin\theta$$

$$\Rightarrow \frac{1}{2}\frac{R^2\omega^2(M+m)}{mg\sin\theta} = S$$

$$\Rightarrow S = \frac{1}{2}\times\frac{0.2\times0.2\times10\times10(2+0.5)}{0.5\times9.8\times\sin30°} = 1.22\text{ m}$$

11. During collision, the torque of the system about P will be zero because the only force acting on the system is through P (namely weight of rods/mass m/reaction at P)



Given : $\ell = 0.6$ m
$m_A = 0.01$ kg
$m_B = 0.02$ kg
$m = 0.05$ kg

Since, $\tau = \frac{dL}{dt}$ and $\tau = 0$

$\Rightarrow$ L is constant.

Angular momentum before collision $= mv \times 2\ell$ ... (i)
Angular momentum after collision $= I\omega$ ... (ii)
Where I is the moment of inertia of the system after collision about P and ω is the angular velocity of the system.

$M.I.$ about $P : I_1 = M.I.$ of mass m
$I_2 = M.I.$ of rod m_A
$I_3 = M.I.$ of rod m_B
$I = I_1 + I_2 + I_3$

$$= \left[m(2\ell)^2 + \left\{m_A\left(\frac{\ell^2}{12}\right) + \left(\frac{\ell}{2}\right)^2\right\} + \left\{m_B\left(\frac{\ell^2}{12}\right) + \left(\frac{\ell}{2}+\ell\right)^2\right\}\right]$$

$$= \left[4m\ell^2 + m_A\left(\frac{\ell^2}{12}+\frac{\ell^2}{4}\right) + m_B\left(\frac{\ell^2}{12}+\frac{9\ell^2}{4}\right)\right]$$

$$= \left[4m\ell^2 + \frac{1}{3}m_A\ell^2 + \frac{7}{3}m_B\ell^2\right] = 0.09\text{ kg } m^2$$

From (i) and (ii)

$$I\omega = mv \times 2\ell$$

$$\Rightarrow \omega = \frac{mv\times2\ell}{I} = \frac{0.05\times v\times2\times0.6}{0.09} = 0.67\,v$$

Applying conservation of mechanical energy after collision.
(Using the concept of mass)
Loss of $K.E.$ = Gain in $P.E.$

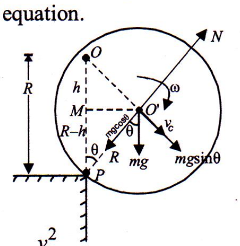
$$\frac{1}{2}I\omega^2 = mg(2\ell) + m_A\left(\frac{\ell}{2}\right)g + m_Bg\left(\frac{3\ell}{2}\right)$$

$$\Rightarrow \frac{1}{2}\times0.09\times(0.67v)^2$$

$$= \left[0.05\times2 + 0.01\times\frac{1}{2} + 0.02\times\frac{3}{2}\right]\times9.8\times0.6$$

$$\Rightarrow v = 6.3\text{ m/s}$$

12. **(a)** Let the original position of centre of mass of the cylinder be O. While rolling down off the edge, let the cylinder be at such a position that its centre of mass is at a position O'. Let $\angle NPO$ be θ. As the cylinder is rolling, the c.m. rotates in a circular path. The centripetal force required for the circular motion is given by the equation.



$$mg\cos\theta - N = \frac{mv_c^2}{R}$$

Where N is the normal reaction and m is mass of cylinder.
The condition for the cylinder leaving the edge is $N = 0$

$$mg\cos\theta = \frac{mv_c^2}{R} \Rightarrow \cos\theta = \frac{v_c^2}{Rg} \quad \text{... (i)}$$

Applying energy conservation from O to O'.
Loss of potential energy of cylinder
= Gain in translational $K.E.$ + Gain in rotational $K.E.$

From (ii), (iii) and (iv), we get

$$mgh = \frac{1}{2}mv_c^2 + \frac{1}{2}\times\frac{1}{2}mR^2\times\frac{v_c^2}{R^2}$$

$$\Rightarrow gh = \frac{1}{2}v_c^2 + \frac{1}{4}v_c^2 = \frac{3}{4}v_c^2 \Rightarrow v_c^2 = \frac{4gh}{3}$$

In $\Delta O'MP$, $\cos\theta = \frac{R-h}{R}$

$$\Rightarrow h = R(1-\cos\theta)$$

$$\therefore \quad v_c^2 = \frac{4g}{3}R(1-\cos\theta) \quad \text{... (v)}$$

From (i) and (v), we get

$$\cos\theta = \frac{4gr}{3Rg}(1-\cos\theta)$$

$$\Rightarrow 3\cos\theta = 4 - 4\cos\theta \Rightarrow \cos\theta = \frac{4}{7}$$

(b) From (v) speed of C.M. of cylinder before leaving contact with edge.

$$v_c^2=\frac{4gR}{3}\left(1-\frac{4}{7}\right)=\frac{4gR}{7}\Rightarrow v_c=\sqrt{\frac{4gR}{7}}$$

(c) Before the cylinder's c.m. reaches the horizontal line of the edge, it leaves contact with the edge as

$$\theta=\cos^{-1}\frac{4}{7}=55.15^\circ$$

Therefore the rotational *K.E.*, which the cylinder gains at the time of leaving contact with the edge remains the same in its further motion. Thereafter the cylinder gains translational *K.E.*

Again applying energy conservation from *O* to the point where c.m. is in horizontal line with edge

$$mgR=\frac{1}{2}I\omega^2+\frac{1}{2}m(v'_c)^2$$

$$mgR=\frac{1}{2}\times\frac{1}{2}mR^2\times\left(\sqrt{\frac{4g}{7R}}\right)^2+\frac{1}{2}m(v'_c)^2$$

$$\because\quad \omega=\frac{v_c}{R}=\sqrt{\frac{4gR/7}{R}}$$

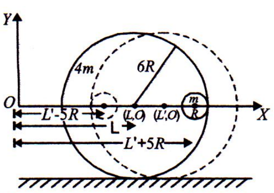
$$\Rightarrow\ mgR-\frac{mgR}{7}=\text{Translational } K.E.=\frac{6mgR}{7}$$

Also, Rotational $K.E.=\frac{1}{2}I\omega^2=\frac{mgR}{7}$

$$\therefore\quad \frac{\text{Translational } K.E.}{\text{Rotational } K.E.}=6$$

13. **KEY CONCEPT :** The concept of center of mass can be applied in this problem.
When small sphere *M* changes its position to other extreme position, there is no external force in the horizontal direction. Therefore the x-coordinate of c.m. will not change.

$$[x_{c.m.}]_{initial}=[x_{c.m.}]_{final}$$



Thin line of sphere represents initial state, dotted line of sphere represents final state.

From (i)

$$(x_{c.m.})_{initial}=(x_{c.m.})_{final}$$

$$\Rightarrow\ \frac{M_1x_1+M_2x_2}{M_1+M_2}=\frac{M_1x'_1+M_2x'_2}{M_1+M_2}$$

$$\Rightarrow\ \frac{4m\times L+m\times(5R+L)}{4m+m}=\frac{4m\times L'+m\times(L'-5R)}{4m+m}$$

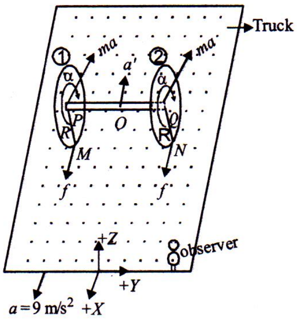
$\Rightarrow\ 5L+5R=5L'-5R$

$\Rightarrow\ 5L+10R=5L' \qquad \Rightarrow\ L+2R=L'$

Since, the individual center of mass of the two spheres has a *y* co-ordinate zero in its initial state and its final state therefore the *y*-coordinate of c.m. of the two sphere system will remain zero.

Therefore the coordinate of c.m. of bigger sphere is $(L+2R, 0)$.

14. **(i)** The observer, let us suppose, is on the accelerated frame. Therefore a pseudo force *ma* is applied individually on each disc on the centre of mass. The frictional force is acting in the + *X* direction which is producing an angular acceleration α.



The torque acting on the disc is

$$\tau=I\alpha=f\times R$$

$$\Rightarrow\ f=\frac{I\alpha}{R}\quad ...(i)$$

Let *a'* is the acceleration of c.m. of the disc as seen by the observer. Since the case is of pure rolling and from the perspective of the observer

$a'=\alpha R$ ...(ii)

$\Rightarrow$ From (i) and (ii)

$$f=\frac{Ia'}{R^2}\quad ...(iii)$$

Applying Newton's law for motion in *X*-direction

$$ma-f=ma'$$

$$\Rightarrow a'=\left(a-\frac{f}{m}\right)\quad ...(iv)$$

Also moment of inertia

$$I=\frac{1}{2}mR^2\quad ...(v)$$

From (iii), (iv) and (v)

$$f=\frac{1}{2}\frac{mR^2\left(a-\frac{f}{m}\right)}{R^2}\Rightarrow 2f=ma-f$$

$$\Rightarrow\ 3f=ma\Rightarrow f=\frac{ma}{3}=\frac{2\times 9}{3}=6\text{N}\quad(\text{In }+X\text{ direction})$$

$$\vec{f}=(6\hat{i})\text{N}$$

(ii) The position vector of point *M*, taking *O* as the origin

$\vec{r_m}=-0.1\hat{j}-0.1\hat{k}$ and position vector of point *N*

$\vec{r_N}=0.1\hat{j}-0.1\hat{k}$

The torque due to friction on disc 1 about *O*

$$\vec{\tau_1}=\vec{r_M}\times\vec{f}=(-0.1\hat{j}-0.1\hat{k})\times(6\hat{i})$$

$$=0.6(\hat{k}-\hat{j})N-m$$

The torque due to friction on disc 2 about *O*

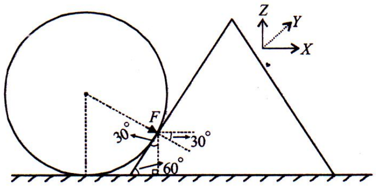
$$\vec{\tau_2}=\vec{r_N}\times\vec{f}=(+0.1\hat{j}-0.1\hat{k})\times(6\hat{i})$$

$$=0.6(-\hat{j}-\hat{k})N-m$$

The magnitude of torque on each disc

$$|\tau_1|=|\tau_2|=0.6\sqrt{2}\,N-m$$

15. **(a)**



Resolving the force F acting on the wedge

$F_x = F\cos 30°;\ F_y = F\sin 30°$

Note : The collision is elastic and since the sphere is fixed, the wedge will return back with the same velocity (in magnitude).

The force responsible to change the velocity of the wedge in X-direction is F_x.

$F_x \times \Delta t = mv - (-mv)$

(Impulse) = (Change in momentum)

$$\therefore\ F_x = \frac{2mv}{\Delta t} \Rightarrow F\cos 30° = \frac{2mv}{\Delta t} \Rightarrow F = \frac{4mv}{\sqrt{3}\,\Delta t}$$

In vector terms

$$\vec{F} = F_x\hat{i} + F_y(-\hat{k}) = F\cos 30°\hat{i} + F\sin 30°(-\hat{k})$$

$$= F\times\frac{\sqrt{3}}{2}\hat{i} + F\times\frac{1}{2}(-\hat{k})$$

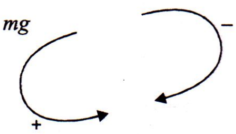
$$\Rightarrow \vec{F} = \frac{F}{2}(\sqrt{3}\hat{i} - \hat{k}) = \frac{2mv}{\sqrt{3}\Delta t}(\sqrt{3}\hat{i} - \hat{k})$$

Taking equilibrium of force in Z-direction (acting on wedge) we get

$F_y + mg = N$

$$\Rightarrow N = \frac{F}{2} + mg = \frac{2mv}{\sqrt{3}\Delta t} + mg$$

$$N = \left(\frac{2mv}{\sqrt{3}\Delta t} + mg\right)\hat{k}$$

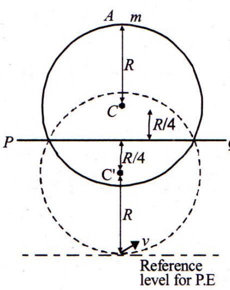


(b) Taking torques on wedge about the c.m. of the wedge.

$F \times h$ – Torque due to $N + mg \times 0 = 0$

$$\Rightarrow \text{Torque due to } N = F\times h = \frac{4mv}{\sqrt{3}\Delta t}\times h$$

16. **KEY CONCEPT :** During the fall, the disc-mass system gains rotational kinetic energy. This is at the expense of potential energy.



Applying energy conservation

Total energy initially = total energy finally

$$mg\left(2R + \frac{2R}{4}\right) + mg\left(R + \frac{2R}{4}\right) = mgR + \frac{1}{2}I\omega^2$$

Where $I = M.I.$ of disc-mass system about PQ

$$mg\times\frac{10R}{4} + mg\frac{6R}{4} = mgR + \frac{1}{2}I\omega^2 \Rightarrow 3mgR = \frac{1}{2}I\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{6mgR}{I}} \qquad \text{...(i)}$$

$$(I)_{PQ} = (I_{disc})_{PQ} + (I_{mass})_{PQ}$$

$$= \left[\frac{mR^2}{4} + M\left(\frac{R}{4}\right)^2\right] + m\left(\frac{5R}{4}\right)^2$$

$$[\because M.I. \text{ of disc about diameter} = \frac{1}{4}MR^2]$$

$$= \frac{mR^2[4+1+25]}{16} = \frac{15mR^2}{8} \qquad \text{...(ii)}$$

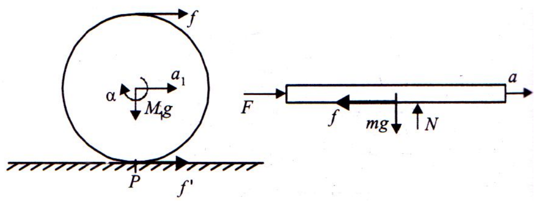
From (i) and (ii)

$$\omega = \sqrt{\frac{6mgR\times 8}{15mR^2}} = \sqrt{\frac{16g}{5R}}$$

Let v be the velocity of mass m at the lowest point of rotation

$$v = \omega\left(R + \frac{R}{4}\right) \quad \therefore\ v = \sqrt{\frac{16g}{5R}}\times\frac{5R}{4} = \sqrt{5gR}$$

17. The man applies a force F in the horizontal direction on the plank as shown. Therefore the point of contact of the plank with the cylinder will try to move towards right. Therefore the friction force F will act towards left on the plank. To each and every action there is equal and opposite reaction. Therefore a frictional force f will act on the top of the cylinder towards right.



Direction of f' : A force f is acting on the cylinder. This force is trying to move the point of contact P towards right by an acceleration

$$a_{cm} = \frac{f}{M_1} \text{ acting towards right.}$$

At the same time, the force f is trying to rotate the cylinder about its centre of mass.

$f \times R = I \times \alpha$

$$\Rightarrow \alpha = \frac{f\times R}{I} = \frac{f\times R}{\frac{1}{2}M_1R^2} = \frac{2f}{M_1R} \text{ in clockwise direction.}$$

$$\therefore\ \alpha_{cm} + \alpha R = \frac{f}{M_1} - \frac{2f}{M_1R}\times R = -\frac{f}{M_1}, \text{ i.e., towards left.}$$

Therefore, the point of contact of the cylinder with the ground move towards left. Hence friction force acts towards right on the cylinder.

Note : You can assume any direction of friction at the point of contact and solve the problem. If the value of friction comes out to be positive, our assumed direction is correct otherwise the direction of friction is opposite. The above activity is done so that if only the direction of friction is asked, an approach may be developed.

Applying Newton's law on plank, we get

$$F - f = m_2 a_2 \quad \text{...(i)}$$

Also, $a_2 = 2a_1$ ...(ii)

Because a_2 is the acceleration of topmost point of cylinder and there is no slipping.

Applying Newton's law on cylinder

$$M_1 a_1 = f + f' \quad \text{...(iii)}$$

The torque equation for the cylinder is

$$f \times R - f' \times R = I\alpha = \frac{1}{2} M_1 R^2 \times \left(\frac{a_1}{R}\right)$$

$$[\because I = \frac{1}{2} M_1 R^2 \text{ and } R\alpha = a_1]$$

$$\therefore \quad (f - f')R = \frac{1}{2} M_1 R a_1 \Rightarrow f + f' = \frac{1}{2} M_1 a_1 \quad \text{...(4)}$$

Solving equation (iii) and (iv), we get

$$f = \frac{3}{4} M_1 a_1 \quad \text{...(5)}$$

$$\text{and } f' = \frac{1}{4} M_1 a_1 \quad \text{...(6)}$$

From (i) and (iii)

$$F - f = 2m_2 a_1 \Rightarrow F - \frac{3}{4} M_1 a_1 = 2m_2 a_1$$

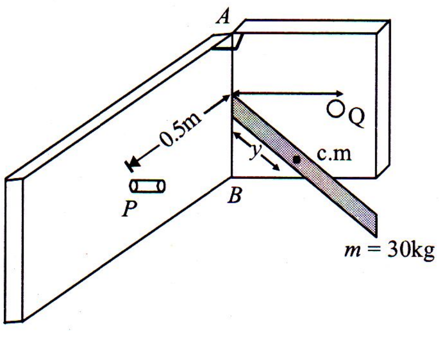
$$\therefore \quad a_1 = \frac{4F}{3M_1 + 8m_2} \quad \therefore a_2 = \frac{8F}{3M_1 + 8m_2}$$

From (v) and (vi)

$$f = \frac{3}{4} M_1 \times \frac{4F}{3M_1 + 8m_2} = \frac{3FM_1}{3M_1 + 8m_2}$$

$$\text{And } f' = \frac{1}{4} M_1 \times a_1 = \frac{FM_1}{3M_1 + 8m_2}$$

18. $I_c = 1.2\ \text{kg-}m^2$



Let y be the distance of c.m. from line AB.

Applying parallel axis theorem of M.I. we get M.I. of laminar sheet about AB

$$I_{AB} = I_{c.m.} + my^2$$

$$I_{AB} = 12 + 30y^2 \quad \text{...(i)}$$

The angular velocity of the laminar sheet will change after every impact because of impulse.

Impulse = Change in linear momentum

$$6 = 30(V_f - V_i)$$

$$6 = 30 \times y(\omega_f - \omega_i) \quad \text{...(ii)}$$

Also, change in angular momentum = Moment of Impulse

$$\therefore \quad I_{AB}\omega_f - I_{AB}\omega_i = \text{Impulse} \times \text{distance}$$

$$I_{AB}(\omega_f - \omega_i) = 6 \times 0.5 = 3$$

$$\therefore \quad \omega_f = \frac{3}{I_{AB}} + \omega_i = \frac{3}{1.2 + 30y^2} + (-1) \quad \text{...(iii)}$$

Note : Minus sign with ω_i because the direction of laminar plate towards the obstacle is taken as – ve (assumption).

From (ii) and (iii)

$$6 = 30 \times y\left[\frac{3}{1.2 + 30y^2} - 1 + 1\right]$$

$$1 = 5y\left[\frac{3}{1.2 + 30y^2}\right]$$

$$\therefore \quad 1.2 + 30y^2 = 5y[+3] = 15y$$

$$\therefore \quad 30y^2 - 15y - 1.2 = 0$$

On solving, we get $y = 0.1$ or 0.4

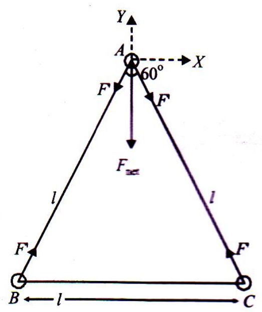
$\therefore \quad \omega_f = 1$ rad/s if we put $y = 0.1$ in eq. (ii)

And $\omega_f = 0.5$ rad/s if we put $y = 0.4$ in eq. (ii)

(Not valid as per sign convention)

Now, since the lamina sheet comes back with same angular speed as that of incident angular speed, the sheet will swing in between P and Q infinitely.

19. **(a)** The mass B is moving in a circular path centred at A. The centripetal force $(m\ell\omega^2)$ required for this circular motion is provided by F'. Therefore a force F' acts on A (the hinge) which is equal to $m\ell\omega^2$. The same is the case for mass C. Therefore the net force on the hinge is



$$F_{net} = \sqrt{F'^2 + F'^2 + 2F'F'\cos 60°}$$

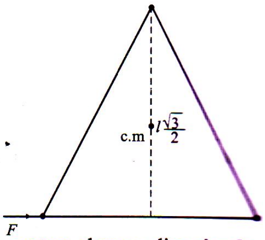
$$F_{net} = \sqrt{2F'^2 + 2F'^2 \times \frac{1}{2}} = \sqrt{3}F' = \sqrt{3}\, m\ell\omega^2$$

(b) The force F acting on B will provide a torque to the system. This torque is

$$F \times \frac{\ell\sqrt{3}}{2} = I\alpha$$

$$F \times \frac{\sqrt{3}\,\ell}{2} = (2m\ell^2)\alpha$$

$$\Rightarrow \alpha = \frac{\sqrt{3}}{4} \times \left(\frac{F}{m\ell}\right)$$



The total force acting on the system along x-direction is

$F + (F_{net})_x$

This force is responsible for giving an acceleration a_x to the system.

Therefore,

$$F + (F_{net})_x = 3m\,(a_x)_{c.m.}$$

$$= 3m\,\frac{F}{4m} \qquad \left(\because\ \alpha_x = \alpha r = \frac{\sqrt{3}}{4}\frac{F}{m\ell} \times \frac{\ell}{\sqrt{3}} = \frac{F}{4}\right)$$

$$= \frac{3F}{4} \qquad \therefore\ (F_{net})_x = -\frac{F}{4}$$

$(F_{net})_y$ remains the same as before $= \sqrt{3}\,m\ell\omega^2$.

20. We know that $\vec{\tau} = \frac{d\vec{L}}{dt}$

$\Rightarrow\ \vec{\tau} \times dt = d\vec{L}$

When angular impulse $(\vec{\tau} \times d\vec{t})$ is zero, the angular momentum is constant. In this case for the wooden log-bullet system, the angular impulse about O is constant.

Therefore,

[angular momentum of the system]$_{initial}$

= [angular momentum of the system]$_{final}$

$\Rightarrow\ mv \times L = I_0 \times \omega$ ... (i)

where I_0 is the moment of inertia of the wooden log-bullet system after collision about O

$I_0 = I_{wooden\ log} + I_{bullet}$

$= \frac{1}{3}ML^2 + ML^2$ ... (ii)

From (i) and (ii)

$$\omega = \frac{mv \times L}{\left[\frac{1}{3}ML^2 + mL^2\right]}$$

$$\Rightarrow\ \omega = \frac{mv}{\left[\frac{ML}{3} + mL\right]} = \frac{3mv}{(M + 3m)L}$$